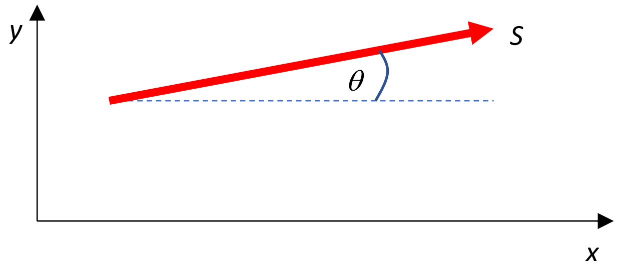
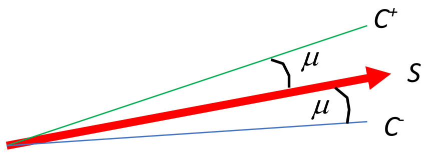
Method of Characteristics (Isentropic Flows)

The Ideas behind the method

Consider a streamline *S* at an angle θ to the *x* direction as shown in the diagram below:



At each point along this streamline, two Mach lines emanate at an angle μ. These are labelled C^+ and C^- in the diagram below:



If *M* is the Mach number of the streamline, then:

$$\mu = sin^{-1}\left(\frac{1}{M}\right)$$

These two Mach lines (C^+ and C^-) are called the *characteristic lines*.

The initial slope of these lines (although over a longer length they're curved) is given by:

$$\left(\frac{dy}{dx}\right)_{C^+/C^-} = tan(\theta \mp \mu)$$

Now, along these characteristic lines, it can be shown that:

$$\theta + \nu(M) = a\ constant$$

Where $\nu(M)$ is the *Prandtl-Meyer function*, which depends on the Mach number *M*. This is tabulated in Appendix C of "Fundamentals of Aerodynamics" (Anderson) or given by the equation:

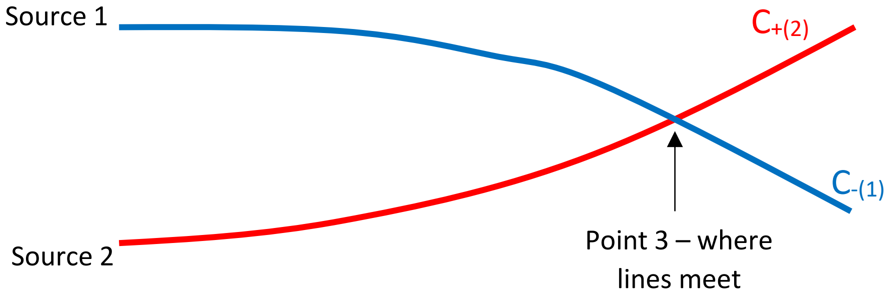
$$\nu(M) = \sqrt{\frac{\gamma+1}{\gamma-1}}tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}(M^2-1)} - tan^{-1}\sqrt{M^2-1}$$

The two constants thus generated are labelled K_- and K_+ as shown overleaf.

$$\theta + \nu(M) = K_- \quad (along\ the\ C_-\ characteristic\ line)$$

$$\theta - \nu(M) = K_+ \quad (along\ the\ C_+\ characteristic\ line)$$

Now if we have two characteristic lines from different sources, we can calculate from this the properties of the point (labelled 3 in the diagram below) where they meet:



$$\theta_3 = \frac{K_{-(1)} + K_{+(2)}}{2}$$

$$\nu(M)_3 = \frac{K_{-(1)} - K_{+(2)}}{2}$$

And from *ν(M)* calculate or look up the Mach number at point 3 – once we know this, all the other parameters are given by the standard isentropic equations.

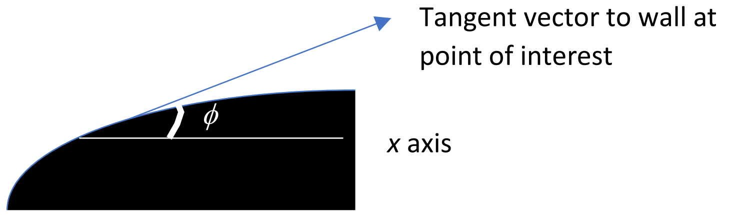
You can find out the point of intercept (providing it's close to the initial points) by assuming that the lines are straight (an OK assumption over a short distance) by using the equations (derived from the previous ones):

$$Slope\ angle\ of\ line\ C_-from\ source\ 1 = \frac{\theta_1 + \theta_3}{2} - \frac{\mu_1 + \mu_3}{2}$$

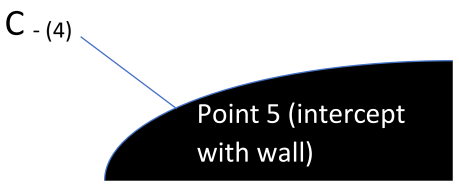
$$Slope\ angle\ of\ line\ C_+from\ source\ 2 = \frac{\theta_2 + \theta_3}{2} + \frac{\mu_2 + \mu_3}{2}$$

We can also work out what happens when the lines reach a wall.

Firstly, we need to know the angle of the wall ϕ as show below:



Now let's say we know the *K* value for a characteristic line (let's say from a source labelled 4) intercepting the wall at the point above (which we'll call point 5):



Now *K* is:

$$K_{-(4)} = \theta_4 + \upsilon(M)_4$$

And we can readily work out the values at point 5 from:

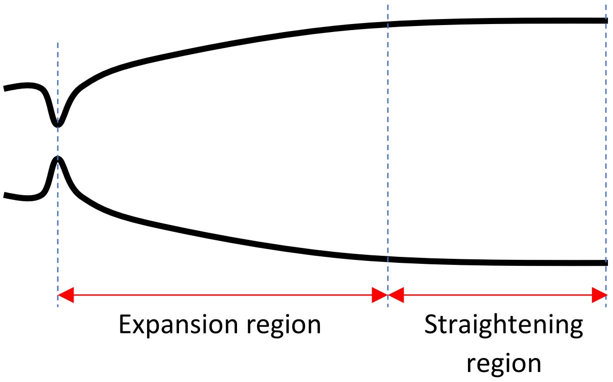
$$K_{-(4)} = K_{-(5)} = \phi + \nu(M)_5$$

So:

$$\nu(M)_5 = \nu(M)_4 + \theta_4 - \phi$$

Application to nozzle flow

First some preliminaries:



A typical "Bell Shaped" Nozzle.

The expansion region is easy to design – there are only expansion waves present - and in fact a straight cone shape is sometimes even used (and some nozzles only have an expansion region with no straightening part). The most commonly shape for the expansion region is probably a section of large radius circle though – as this an easier shape to attach a straightening section to.

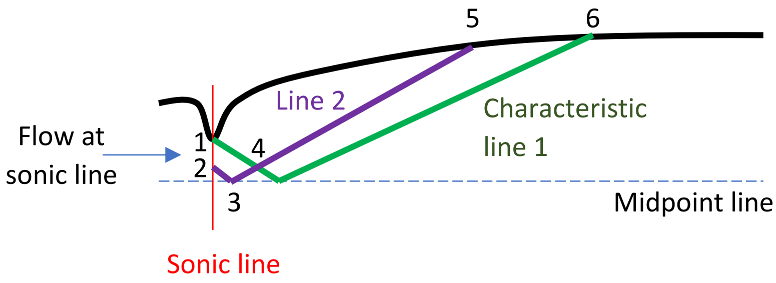
The purpose of the straightening region is to better match the outlet pressure of the nozzle to ambient (and, in the case of a wind tunnel, to provide a suitable longer test-section).

It is in the straightening region where problems occur – as the shape is now turning in on itself and shockwaves may therefore form. The idea is to design the straightening section so that the resulting shockwaves cancel themselves out and the flow continues to be isentropic.

A suitably designed shape including both expansion and straightening sections is commonly referred to as a "bell shape". Note however, that it has been shown that certain parabola and other shapes are good approximations to the bell shape and don't require the in-depth design.

The method of characteristics can be used to analyse and design the straightening section of bell nozzle.

To use the method, only half the nozzle is required (because it is symmetric). We also know the conditions at the throat (and the direction of the stream-lines, because they are perpendicular to the sonic line)



Points on diagram:

Point 1 – source of characteristic line 1 (at edge of throat).

Point 2 – source of characteristic line 2 (at other point further down the sonic line).

Point 3 – line 2 meets central axis (and also meets symmetrical line from the lower half of the nozzle coming in the opposite direction - and so appears to have been reflected).

Point 4 - where lines 2 and 3 cross.

Point 5 - where line 2 reaches nozzle contour wall.

Point 6 - where line 1 reaches nozzle contour wall.

Analysis:

1. Line (2) from point 2 intersects centreline at point 3:

$$\theta_3 + \upsilon_3 = K_{-(3)}$$

But:

$$\theta_3 = 0$$

Because the flow at point 3 is straight along the centreline of the nozzle. And:

$$K_{-(3)} = K_{-(2)}$$

And so we have $\nu(M)_3$

2. Because we know the properties from points 3 and 1. So we can calculate 4.

3. At point 5 the wall angle ϕ is known, so the other properties can be calculated.

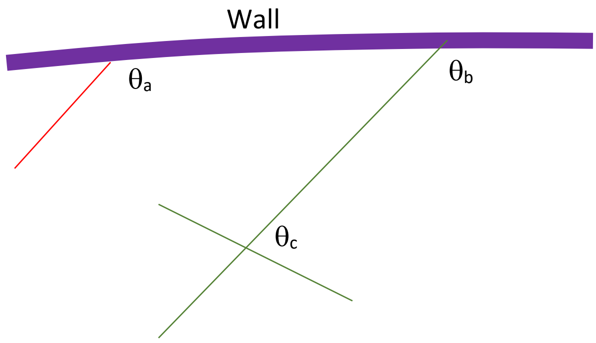
4. Propagate the other points (of your choosing) along the expanding section in the same way.

5. Lines ending up in the straightening region don't reflect (the contour is turning into the flow). Lines terminating here (for example point 6 in the diagram on the previous page) are drawn straight from the last point crossed and have the parameters of that point.

Design of the straightening region curve

To design the curve of the straightening section, first propagate all the lines through the expansion section as described in the previous section.

As mentioned in last point of the previous section the parameters of the wall intersecting line are determined by the last known point crossed. This is illustrated in the diagram below:



So, angle θ_b is the same as that at the last calculated point θ_c that is $\theta_b = \theta_c$ and the wall angle (the slope of the wall) between two calculated angles like θ_a and θ_b is:

$$\theta_{wall} = \frac{\theta_a + \theta_b}{2}$$

References

Most aerodynamics books cover the method of characteristics, but there are particularly in-depth explanations in John D Anderson's books: "Fundamentals of Aerodynamics", "Modern Compressible Flow" and "Computational Fluid Dynamics"